

Time Dependent Supersymmetry in Quantum Mechanics ¹

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Abstract

The well-known supersymmetric constructions such as Witten's supersymmetric quantum mechanics, Spiridonov–Rubakov parasupersymmetric quantum mechanics, and higher-derivative SUSY of Andrianov et al. are extended to the nonstationary Schrödinger equation. All these constructions are based on the time-dependent Darboux transformation. The superalgebra over the conventional Lie algebra is constructed. Examples of time-dependent exactly solvable potentials are given.

1. The supersymmetric quantum mechanics is in a permanent intensive development since the Witten papers [1]. One can cite the N -extended supersymmetric quantum mechanics [2], parasupersymmetric quantum mechanics [3], and higher order derivative supersymmetric quantum mechanics [4]. The field of supersymmetric quantum mechanics is recently reviewed in [5]. We want to point out that all above mentioned constructions are valid for the time independent Hamiltonians and if one restrict oneself by the stationary solutions of the Schrödinger equation. Hence, these constructions can be referred to the stationary supersymmetric quantum mechanics and the nonstationary one needs to be developed. We hope that this report gives a stimulus for the further developments in this domain.

2. The nonstationary supersymmetric quantum mechanics is based on the nonstationary Darboux transformation [6] just in the same way as the stationary one [1, 7] is based on the conventional Darboux transformation [8].

Let us consider two time-dependent Schrödinger equations

$$(i\partial_t - H_0)\psi(x, t) = 0, \quad \partial_t = \partial/\partial t, \quad H_0 = -\partial_x^2 - V_0(x, t), \quad \partial_x^2 = \partial_x \partial_x, \quad (1)$$

$$(i\partial_t - H_1)\varphi(x, t) = 0, \quad H_1 = -\partial_x^2 - V_1(x, t), \quad x \in R, \quad t \in \mathbb{R}^1. \quad (2)$$

Here $-V_0(x, t)$ is a potential energy and $R = [a, b]$ is the interval for variable x which can be both finite and infinite. If the Schrödinger operators for Eqs. (1)

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and (2) are related by the intertwining relation

$$L(i\partial_t - H_0) = (i\partial_t - H_1)L, \quad (3)$$

where L is a linear operator, called *transformation operator*, the functions ψ and φ are related as follows: $\varphi = L\psi$ if $L\psi \neq 0$.

If $(i\partial_t - H_0)$ and $(i\partial_t - H_1)$ are self-adjoint (in the sense of some scalar product) the equation (3) implies

$$L^+(i\partial_t - H_1) = (i\partial_t - H_0)L^+, \quad (4)$$

where the superscript plus sign $(+)$ is used to denote the operator adjoint to L , and Eqs. (1) and (2) become “peer”. It follows from Eqs. (3) and (4) that $s_0 = L^+L$ commutes with $(i\partial_t - H_0)$ and $s_1 = LL^+$ commutes with $(i\partial_t - H_1)$ and consequently s_0 is a symmetry operator for the initial equation (1) and s_1 is a symmetry operator for the final one (2).

The constructions such as in Eq. (3) are well-known in mathematics and are intensively investigated since the Delsart’s paper [9]. The most significant results obtained with the help of the transformation operators concern the backscattering problem in quantum mechanics [10] and its application for the solving of the nonlinear equations [11].

3. We now assume operator L to be a differential of the first degree in ∂_x with smooth coefficients depending on both variables x and t . We should not include in L the derivative ∂_t since it, being found from equation (1), transforms L into the second-order differential operator. In this case the operator L and the real potential difference $A(x, t) = V_1(x, t) - V_0(x, t)$ are completely defined by a function $u(x, t)$ called *transformation function* [6]:

$$L = L_1(-u_x/u + \partial_x), \quad (5)$$

$$L_1 = L_1(t) = \exp(2 \int dt \operatorname{Im}(\log u)_{xx}), \quad (6)$$

$$A = (\log |u|^2)_{xx}. \quad (7)$$

To obtain a real potential difference we should impose the reality condition [6] on the function u

$$(\log u/u^*)_{xxx} = 0, \quad (8)$$

where the asterisk implies the complex conjugation.

In the majority of cases of physical interest we can introduce the Hilbert space structure $L_0^2(R)$ in the space of solutions of the equation (1) with the scalar product appropriately defined. Symmetry operator $s_0 = L^+L$, being symmetric, may be extended up to self-adjoint in the appropriate Hilbert space and can have either discrete spectrum or continuous one. Since $Lu = 0$ (see Eq. (5)), the

function u is an eigen function of this operator corresponding to zero eigen value. Hence, u is one of the eigen functions of operator $L^+L = h - \alpha$. In general, h is a self-adjoint integral of motion for the initial Schrödinger equation (1) which in particular case (if V_0 does not depend on t) may be equal to the Hamiltonian H_0 .

With the help of the transformation operator L , just in the same way that in the conventional supersymmetric quantum mechanics [1, 7], we can construct the supercharge operators

$$Q = \begin{pmatrix} 0 & 0 \\ L & 0 \end{pmatrix} = (Q^+)^{\dagger} \quad (9)$$

acting on two-component wavefunctions $\Psi(x, t) = \begin{pmatrix} \psi(x, t) \\ L\psi(x, t) \end{pmatrix}$. Two Schrödinger equations (1) and (2) may now be rewritten in supersymmetric form

$$(iI\partial_t - \mathcal{H})\Psi(x, t) = 0, \quad (10)$$

where I is 2×2 identity matrix and $\mathcal{H} = \begin{pmatrix} H_0 & 0 \\ 0 & H_1 \end{pmatrix}$ is a superhamiltonian. Since $s_0 = L^+L$ and $s_1 = LL^+$ are symmetry operators for equations (1) and (2) respectively, the superoperator $S = \begin{pmatrix} L^+L & 0 \\ 0 & LL^+ \end{pmatrix}$ is the symmetry operator for the equation (10). The operators Q , Q^+ , and S form a well-known superalgebra [1, 7]. There is a single difference, namely, instead of the Hamiltonians we see other integrals of motion of the equation (1) in its construction. When h is the initial Hamiltonian above constructions coincide with known ones. This is the reason to call the transformation (5), (6), *time-dependent Darboux transformation* [6].

4. With the help of the other eigen functions of operator h we can perform the chain of Darboux transformations and construct the parasuperalgebra in full analogy with papers [3]. If in this chain we eliminate all intermediate operators h and express the final operator L in terms of particular solutions u_i of the initial equation (1), we obtain higher-derivative nonstationary quantum mechanics similar to the stationary one [4]. In this case

$$L \equiv L^{(N)} = L_N(t)W^{-1}(u_1, \dots, u_N) \begin{vmatrix} u_1 & u_2 & \dots & 1 \\ u_{1x} & u_{2x} & \dots & \partial_x \\ \vdots & & & \\ u_{1x}^{(N)} & u_{2x}^{(N)} & \dots & \partial_x^N \end{vmatrix} \quad (11)$$

where W stands for the conventional symbol of the Wronskian of the functions

u_1, \dots, u_N . For the real function $L_N(t)$ we have

$$L_N(t) = \exp \left\{ 2 \int dt \operatorname{Im} [\log W(u_1, \dots, u_N)]_{xx} \right\}. \quad (12)$$

The reality condition (8) takes now the form

$$\left[\log \frac{W(u_1, \dots, u_N)}{W^*(u_1, \dots, u_N)} \right]_{xxx} = 0. \quad (13)$$

For the potential difference we obtain

$$A_N = (\log |W(u_1, \dots, u_N)|^2)_{xx}. \quad (14)$$

We can recognise in formula (11) the generalisation of the known Crum-Krein formula [12, 13] to the nonstationary case. Note that the condition (13) is more feeble than the reality condition (8) imposed on every function u_i . Thus, we can construct the higher-derivative supersymmetry with the self-adjoint final Hamiltonian even if the intermediate Hamiltonians are not self-adjoint (so-called irreducible case described for stationary equation in Ref. 4). The basic relation for the time-dependent polynomial supersymmetric quantum mechanics is the following factorisation properties

$$L^+ L = \prod_{i=1}^N (h_0 - C_i), \quad L L^+ = \prod_{i=1}^N (h_1 - C_i) \quad (15)$$

first obtained for the stationary case in Ref. 4. The C_i in Eqs. (15) are eigenvalues corresponding to the eigen functions u_i of the operator h_0 .

5. To obtain a regular potential difference by the Darboux transformation (5)–(7) the transformation function u should be nodeless. In the space $L_0^2(R)$ a single nodeless eigen function of the operator h exists (if it has a discrete spectrum). This function is the ground state function. Beyond the space $L_0^2(R)$ there are many nodeless eigen functions of the operator h suitable for use as the transformation functions. They should have eigen values $\alpha < \varepsilon_0$ (ε_0 being the lowest eigenvalue of h). In this case the discrete spectra of the symmetry operators $h = L^+ L + \alpha$ and $\bar{h} = L L^+ + \alpha$ differ by a single level and we have unbroken supersymmetry [1]. Every bounded state of the superoperator S , except for its ground state, is double degenerate.

We will now describe unexpected peculiarities in the breakdown of the supersymmetry in the higher-derivative supersymmetric quantum mechanics. These peculiarities (as far as we know) are not discussed in the available literature.

The Darboux transformation (5,6) being performed with the discrete spectrum function $u_n(x, t)$ of the operator h [it has n zeros in the interval (a, b)] gives

the potential difference with n poles. Solutions obtained with the help of the transformation operator (5) are not square integrable functions in $[a, b]$. Nevertheless, the second transformation with the transformation function $u_{n+1}(x, t)$, having $n + 1$ zeros in (a, b) removes all singularities and the transformation operator of the second degree $L^{(2)} = L_2 L_1$, where $L_{1,2}$ are the first order Darboux transformation operators, is well defined. This fact reflects the known property of the Wronskian constructed from the functions u_{k_i} belonging to the space $L^2(R)$ [13]: the Wronskian $W(u_{k_1}, \dots, u_{k_N})$ conserves its sign if for all $k = 0, 1, 2, \dots$ the inequality $(k - k_1)(k - k_2) \cdots (k - k_N) \geq 0$ holds. In particular, the functions u_{k_i} may be two by two juxtaposed functions. The discrete eigenvalues α_{k_i} of the operator h are absent in the spectrum of transformed operator \bar{H} . This signifies that the ground state level of the superoperator S is two-fold degenerate and the excited states constructed with the help of the functions u_{k_i} are nondegenerate. Furthermore, these states are annihilated by the operators Q and Q^+ whereas the ground state is annihilated only by the one of these operators. It should be noted that this property remains valid for the stationary case, i.e., in the conventional supersymmetric quantum mechanics.

6. Differential symmetry operators for the stationary Schrödinger equation are Hamiltonian and its polynomial functions. Symmetry algebra of the nonstationary Schrödinger equation is more rich than for the stationary case. We can use this algebra to construct a superalgebra. For this purpose we should define an operator inverse to L .

The equation (5) implies $Lu = 0$. Choose the transformation function u such that the absolute value of $u^{-1}(x, t)$ is square-integrable in the interval R and the condition (8) is satisfied. Then for every $\psi \in L_0^2(R)$ we have $\varphi = L\psi \in L_1^2(R)$, but the set $L_{11}^2(R) = \{\varphi : \varphi = L\psi, \psi \in L_0^2(R)\}$ does not span the whole space $L_1^2(R)$. The function $\varphi_0(x, t) = [L_1(t)u^*(x, t)]^{-1} \in L_1^2(R)$ [6] can not be obtained by the action of the operator L on any $\psi \in L_0^2(R)$. If we designate by $L_{10}^2(R)$ the linear hull of the function φ_0 then $L_1^2(R) = L_{10}^2(R) \oplus L_{11}^2(R)$.

Choose as the transformation function the function $v = \varphi_0$ and define the following integral operator acting from $L_{11}^2(R)$ to $L_0^2(R)$

$$M\varphi(x, t) = [L_1(t)v^*(x, t)]^{-1} \int_a^x v^*(y, t)\varphi(y, t)dy. \quad (16)$$

The straightforward calculation persuades that $LM\varphi = \varphi$ for all $\varphi \in L_{11}^2(R)$ and the condition $v \in L_{10}^2(R)$ implies $ML\psi = \psi$ for all $\psi \in L_0^2(R)$ operator M , hence, is inverse to L : $M = L^{-1}$, and we have one-to-one correspondence between the spaces $L_0^2(R)$ and $L_{11}^2(R)$.

If in the space $L_0^2(R)$ the symmetry operators g_i forming n -dimensional Lie algebra G with the structural constants $f_{ij}^l : [g_i, g_j] = f_{ij}^l g_l$ are defined and this space is invariant under the action of these operators then in the space $L_{11}^2(R)$

we can define the operators $\bar{g}_i = Lg_iM$ and this space will be invariant under the action of all \bar{g}_i . Furthermore these operators form a basis for the n -dimensional Lie algebra \bar{G} with the same structural constants f_{ij}^l . Let T_0 be the space of two-component wave functions $\Psi(x, t)$ with the basis $\Psi_+(x, t) = \psi(x, t)e_+$ and $\Psi_-(x, t) = L\psi(x, t)e_-$, $\psi \in L_0^2(R)$, and $e_+ = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $e_- = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. In the space T_0 we can define operators

$$G_i = \begin{pmatrix} g_i & 0 \\ 0 & \bar{g}_i \end{pmatrix} \quad (17)$$

which form a Lie algebra isomorphic to G . The operators G_i are symmetry operators for the supersymmetric equation (10). Besides the operators G_i in the space T_0 the following operators can be defined: $P_0 = L\sigma^-$, $Q_i = g_iL^{-1}\sigma^+$ where $\sigma^- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, $\sigma^+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. These operators are evidently nilpotent: $P_0^2 = 0$, $Q_i^2 = 0$, and $\{Q_i, Q_j\} \equiv Q_iQ_j + Q_jQ_i = 0$. Furthermore, we can find by the direct calculations that $\{P_0, Q_i\} = G_i$ and the generalised Jacobi identities are fulfilled. The operators G_i, Q_i, P_0 , $i = 1, 2, \dots, n$, hence, form a basis for $2n + 1$ -dimensional Lie superalgebra sG .

Note that since $h = L^+L + \alpha \in G$ we have for the operator S introduced in sec. 3, $S \in sG$ and $Q, Q^+ \in sG$.

7. Examples. Consider first the simplest case of a free particle: $v_0(x, t) = 0$. Choose the following solutions of the initial Schrödinger equation (1) [14]:

$$\begin{aligned} \psi_\lambda(x, t) &= (1 + t^2)^{-1/4} \exp[ix^2t/(4 + 4t^2) + i\lambda \arctan t] Q_\lambda(z), \\ z &= x(1 + t^2)^{-1/2}, \end{aligned} \quad (18)$$

where $Q_\lambda(z)$ is the parabolic cylinder functions satisfying the equation $Q_\lambda''(z) - (z^2/4 + \lambda)Q_\lambda(z) = 0$ with λ being an arbitrary parameter (a separation constant). For $\lambda = n + 1/2$, $n = 0, 1, 2, \dots$, the functions $Q_\lambda(z)$ are expressed via the Hermite polynomials $Q_{n+1/2}(z) = \exp(z^2/4)H_n(iz/\sqrt{2})$. The reality condition (8) is satisfied for all real λ . Functions (18) being nodeless for $\lambda = n + 1/2$ and for even n are suitable for use as transformation functions. Formula (7) gives the new potential

$$v_1^{(2k)}(x, t) = -(1 + t^2)^{-1} \left(1 + 4k(2k - 1) \frac{q_{2k-2}(z)}{q_{2k}(z)} - 8k^2 \left(\frac{q_{2k-1}(z)}{q_{2k}(z)} \right)^2 \right),$$

where $q_k(z) = (-i)^k He_k(iz)$, $He_k(z) = 2^{-k/2} H_k(z/\sqrt{2})$.

The same functions for odd n are nodeless in the semiaxis $(0, \infty)$ and with their help we obtain the following time-dependent exactly solvable potential

$$\begin{aligned} v_1^{(2k+1)} &= -(1 + t^2)^{-1} \left(1 + 4k(2k + 1) \frac{q_{2k-1}(z)}{q_{2k+1}(z)} - 2(2k + 1)^2 \left(\frac{q_{2k}(z)}{q_{2k+1}(z)} \right)^2 \right), \\ x &= z\sqrt{1 + t^2} \in (0, \infty). \end{aligned}$$

The functions (18) for $\lambda = -n - 1/2$ form a discrete basis set in $L_0^2(\mathbb{R}^1)$. The double Darboux transformation with juxtaposed functions $u_n = \psi_{-n-1/2}$ and u_{n+1} gives a regular potential of the form

$$\begin{aligned} v_2^{(n,n+1)}(x, t) &= -2(1 + t^2)[J_n''(z)/J_n(z) - (J_n'(z)/J_n(z))^2 - 1], \\ J_k(z) &= \Gamma(k+1) \sum_{s=0}^k \Gamma^{-1}(s+1) He_s^2(z) = kJ_{k-1}(z) + He_k^2(z), \\ J_0(z) &= 1, \quad J_1(z) = z^2 + 1, \quad J_2(z) = z^4 + 3, \dots \end{aligned}$$

These are the potentials which correspond to the supersymmetric model with two-fold degenerate eigenvalue of the superoperator S except for the eigenvalues corresponding to the functions u_n and u_{n+1} . When $n > 0$ the ground state of S is two-fold degenerate and nondegenerate eigenvalues are situated in the middle of the spectrum of S

It is not difficult to establish that the Wronskian constructed from two functions (18) with $\lambda_1 = m + 1/2$ and $\lambda_2 = l + 1/2$ for $m = 0, 2, 4, \dots$ and $l = m + 1, m + 3, \dots$ is nodeless and therefore these functions are suitable for double Darboux transformation. This gives the following exactly solvable potential

$$\begin{aligned} v_2^{(m,l)}(x, t) &= -2(1 + t^2)^{-1}(1 + d^2 \log f_{ml}(z)/dz^2), \\ f_{ml}(z) &= q_m(z)q_{l+1}(z) - q_l(z)q_{m+1}(z). \end{aligned}$$

We will cite one example for harmonic oscillator potential as well: $H_0 = -\partial_x^2 + \omega^2 x^2$, $H_0 \psi_n = (2n + 1)\Psi_n$, $\psi_n = H_n(\sqrt{\omega}x) \exp(-i\omega(2n + 1)t - \omega x^2/2)$, $n = 0, 1, 2, \dots$. If we choose the following nonstationary solution of the initial Schrödinger equation (1) as the transformation function:

$$\begin{aligned} u(x, t) &= \sin^{-1/2}(2\omega t) \cos h(\lambda x / \sin 2\omega t) \times \\ &\exp[i(\omega x^2 - \lambda^2/\omega) \cot(2\omega t)/2] \notin L_0^2(\mathbb{R}^1), \quad \lambda \in \mathbb{R}^1, \end{aligned}$$

we obtain the nonstationary anharmonic potential of the form: $v_1(x, t) = \omega^2 x^2 - 2\lambda^2 \sin^{-1/2}(2\omega t) \operatorname{sech}^2(\lambda x / \sin 2\omega t)$.

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